On the Numerical Accuracy of First-Order Approximate Solutions to DSGE Models*

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Abstract

Many algorithms that provide approximate solutions for dynamic stochastic general equilibrium (DSGE) models employ the QZ-factorization since it allows for a flexible formulation of the model and exempts the researcher from identifying equations that give raise to infinite eigenvalues. We show, by means of an example, that the policy functions obtained by this approach may differ from both the solution of a properly reduced system and the solution obtained from solving the system of non-linear equations which arises from applying the implicit function theorem to the model's equilibrium conditions. As a consequence, simulation results may depend on the specific algorithm used and on the numeric values of parameters that are theoretically irrelevant. The source of this inaccuracy are ill-conditioned matrices as they emerge, e.g., in models with strong habits. Researchers should be ware of those strange effects, and we propose several ways to handle them.

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1 Introduction

Dynamic stochastic general equilibrium (DSGE) models have become the workhorse of macroeconomic research. While the early proponents of this approch had to write their own computer code, their contemporaneous successors can resort to a variety of freely available toolkits, among which DYNARE is probably the most well-known and most versatile one.[1] The user-friendly toolkits have spurred the further development and prevalence of DSGE models, since they have reduced the barriers for potential users considerably. One does not have to unterstand the details of a particular algorithm, the pitfalls of numerical mathematics, and the subtleties of different programming languages in order to solve, simulate, and even estimate a particular model.

In this paper we argue for a careful use. In particular, we illustrate by means of an example different degrees of numerical accuracy that depend on the particular algorithm to obtain the linear part of the model's approximate solution. Errors that occur at this stage affect the computation of higher order terms of the solution. The example is by no means specific. Rather, versions of this model are routinely employed in studies of the equity premium puzzle.[2]

Our benchmark is the (stable) solution of the system of non-linear equations obtained from applying the implicit function theorem to the model's equilibrium conditions. We compare this solution with those that result from linearizing the equilibrium conditions and from solving the respective stochastic first-order system of difference equations. The prevalent way to do this is to use matrix factorization. We consider the QZ-factorization, and, for a properly reduced system, the Schur decomposition. Theoretically, i.e., ignoring errors from finite precision computer arithmetic, all these algorithms will deliver the same solution. In our application, however, some of the elements of the solution differ remarkably between the various methods. As a consequence, we also observe differences in the second moments of simulated time series.

The researcher who relies on the use of DSGE toolkits, thus, should be aware of those strange effects. We propose several ways to handle them. Euler equation residuals, error bounds for the eigenvalues, and the non-linear equations can be used to detect a potential problem. A reformulation of the model in terms of transformed variables or

[1] Others are the toolkit of Harald Uhlig (1999), and the programs of Sims (2002).

[2] See Heer and Maußner (2013) for an overview of those models.

equations as well as a specific balancing of ill-conditioned matrices are ways to improve accuracy.

From here we proceed with a brief description of the canonical DSGE model, the linearized form of this model, and the matrix factorizations in the next section. Section 3 presents our example. Section 4 concludes. The Appendix covers the details of the model presented in Section 3 and additional information on its approximate solutions.

2 Analytical Framework

2.1 Canonical DSGE Model

Our framework closely follows Schmitt-Grohé and Uribe (2004). The solution based on the QZ-factorization is due to Klein (2000) and the presentation follows Heiberger et al. (2014).

Let $\mathbf{x}_t \in \mathbb{R}^{n(x)}$, $\mathbf{z}_t \in \mathbb{R}^{n(z)}$, and $\mathbf{y}_t \in \mathbb{R}^{n(y)}$ denote a vector of endogenous state variables, exogenous state variables, and not predetermined (jump) variables, respectively. The equilibrium conditions of a dynamic, stochastic general equilibrium (DSGE) model are

$$\mathbf{0}_{n(x)+n(y)} = \mathbb{E}_t \mathbf{g}(\mathbf{x}_t, \mathbf{z}_t, \mathbf{y}_t, \mathbf{x}_{t+1}, \mathbf{z}_{t+1}, \mathbf{y}_{t+1}), \tag{2.1a}$$

$$\mathbf{z}_{t+1} = \Pi \mathbf{z}_t + \sigma \Omega \boldsymbol{\epsilon}_{t+1}, \quad \boldsymbol{\epsilon}_{t+1} \sim \mathcal{N}(\mathbf{0}_{n(z)}, I_{n(z)}), \tag{2.1b}$$

where the operator $\mathbb{E}_t$ denotes expectations as of period t. Perturbation methods yield approximate solutions

$$\mathbf{x}_{t+1} = \mathbf{h}^x(\mathbf{x}_t, \mathbf{z}_t, \sigma), \tag{2.2a}$$

$$\mathbf{y}_t = \mathbf{h}^y(\mathbf{x}_t, \mathbf{z}_t, \sigma). \tag{2.2b}$$

2.2 The Set of Non-Linear Equations

Let $\mathbf{w}_t = [\mathbf{x}_t', \mathbf{z}_t']'$, $\tilde{\Omega} = [\mathbf{0}'_{n(x)\times n(z)}, \Omega']'$ and denote the solution of the model more compactly by

$$\mathbf{w}_{t+1} = \mathbf{h}^w(\mathbf{w}_t, \sigma) + \sigma \tilde{\Omega} \boldsymbol{\epsilon}_{t+1} = \begin{bmatrix} \mathbf{h}^x(\mathbf{w_t}, \sigma) \\ \mathbf{h}^z(\mathbf{w_t}, \sigma) + \sigma \Omega \boldsymbol{\epsilon}_{t+1} \end{bmatrix},$$

$$\mathbf{y}_t = \mathbf{h}^y(\mathbf{w}_t, \sigma).$$

Then by defining

$$\tilde{\mathbf{g}}(\mathbf{w}_t, \mathbf{y}_t, \mathbf{w}_{t+1}, \mathbf{y}_{t+1}) = \begin{bmatrix} \mathbf{g}(\mathbf{x}_t, \mathbf{z}_t, \mathbf{y}_t, \mathbf{x}_{t+1}, \mathbf{z}_{t+1}, \mathbf{y}_{t+1}) \\ \mathbf{z}_{t+1} - \Pi \mathbf{z}_t \end{bmatrix}$$

(2.1) can be written as:

$$\begin{aligned} \mathbf{0}_{n(w)+n(y)} &= \mathbb{E}_t \tilde{\mathbf{g}}(\mathbf{w}_t, \mathbf{h}^y(\mathbf{w}_t, \sigma), \mathbf{h}^w(\mathbf{w}_t, \sigma) + \sigma\tilde{\Omega}\boldsymbol{\epsilon}_{t+1}, \mathbf{h}^y(\mathbf{h}^w(\mathbf{w}_t, \sigma) + \sigma\tilde{\Omega}\boldsymbol{\epsilon}_{t+1}, \sigma)) \\ &= \mathbf{G}(\mathbf{w}_t, \sigma). \end{aligned}$$

According to the implicit function theorem the partial derivatives of $\mathbf{G}$ with respect to $\mathbf{w}_t$ must vanish at the stationary solution $\mathbf{w}$ obtained from $\sigma = 0$. This yields a system of equations in the $(n(w) + n(y)) \times n(w)$ coefficients of the linear part of $\mathbf{h}^w$ and $\mathbf{h}^y$, denoted by L^w and L^y, respectively:

$$\begin{aligned} &0 = \tilde{g}^i_{w_{jt}} + \sum_{l=1}^{n(y)} \tilde{g}^i_{y_{lt}} L^y_{l,j} + \sum_{l=1}^{n(w)} \tilde{g}^i_{w_{lt+1}} L^w_{l,j} + \sum_{l=1}^{n(y)} \tilde{g}^i_{y_{lt+1}} \sum_{k=1}^{n(w)} L^y_{l,k} L^w_{k,j}, \\ &i = 1, \ldots, n(w) + n(y), \\ &j = 1, \ldots, n(w). \end{aligned} \tag{2.3}$$

One must pick the solution of this system for which the eigenvalues of the matrix L^w are within the unit circle so that the linearized dynamic system $\mathbf{w}_{t+1} = \mathbf{w} + L^w(\mathbf{w}_t - \mathbf{w})$ is stable.

2.3 The AB-Model and the QZ factorization

A second way to obtain the matrices L^w and L^y is to linearize the system (2.1) at the point $(\mathbf{x}, \mathbf{0}, \mathbf{y})$ solving $\mathbf{g}(\mathbf{x}, \mathbf{0}, \mathbf{y}, \mathbf{x}, \mathbf{y}, \mathbf{0}) = \mathbf{0}_{n(x)+n(y)}$. This yields the system of linear stochastic difference equations:

$$A\, \mathbb{E}_t \begin{bmatrix} \bar{\mathbf{w}}_{t+1} \\ \bar{\mathbf{y}}_{t+1} \end{bmatrix} = B \begin{bmatrix} \bar{\mathbf{w}}_t \\ \bar{\mathbf{y}}_t \end{bmatrix}, \quad \bar{\mathbf{w}}_t \equiv \begin{bmatrix} \mathbf{x}_t - \mathbf{x} \\ \mathbf{z}_t \end{bmatrix}, \quad \bar{\mathbf{y}}_t \equiv \mathbf{y}_t - \mathbf{y}, \tag{2.4a}$$

$$A = \begin{bmatrix} \mathbf{g}_4 & \mathbf{g}_5 & \mathbf{g}_6 \\ \mathbf{0}_{n(z)\times n(x)} & I_{n(z)} & \mathbf{0}_{n(z)\times n(y)} \end{bmatrix}, \tag{2.4b}$$

$$B = \begin{bmatrix} -\mathbf{g}_1 & -\mathbf{g}_2 & -\mathbf{g}_3 \\ \mathbf{0}_{n(z)\times n(y)} & \Pi & \mathbf{0}_{n(z)\times n(y)} \end{bmatrix}. \tag{2.4c}$$

where $\mathbf{g}_i$ denotes the Jacobian matrix of $\mathbf{g}$ with respect to its i-th argument.

Usually, the linear system (2.4) contains a number of equations that involve only variables dated at time t. These arise, e.g., from equations like the economy's resource constraint or from static first-order conditions. In this case the matrix A is singular so that $A^{-1}B$ does not exist and the procedure outlined by Blanchard and Kahn (1980), cannot be applied.[3] As pointed out by Klein (2000), the QZ-factorization can be used to solve the system (2.4).[4]

There are two ways to use the QZ-factorization to solve the model (2.4a). As shown in Heiberger et al. (2014) both provide the same solution (if it exists at all). The first way (see Klein (2000)) rests on factoring the matrix pencil $(B - \lambda A)$, the second on factoring $(A - \mu B)$ (see Heer and Maußner (2009)). The QZ-factorization of the pencil $(B - \lambda A)$ is:

$$\begin{aligned} Q^H A Z &= S, \\ Q^H B Z &= T, \end{aligned} \tag{2.5}$$

where Q and Z are unitary matrices, S and T are upper triangular matrixes, and Q^H denotes the Hermitian transpose of Q.[5] The eigenvalues of the matrix pencil are given by $\lambda_i = t_{ii}/s_{ii}$ for $s_{ii} \neq 0$.[6] Furthermore, the matrices S and T can be arranged so that the eigenvalues appear in ascending order with respect to their absolute value. Assume that $n(w) = n(x) + n(z)$ eigenvalues have modulus less than one and $n(y)$ have modulus greater than one. Let Z_{11} denote the upper left $n(w) \times n(w)$ block of Z, Z_{12} the upper right $n(w) \times n(y)$ block, etc., and define new variables:

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{w}}_t \\ \tilde{\mathbf{y}}_t \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{w}}_t \\ \bar{\mathbf{y}}_t \end{bmatrix}, \tag{2.6}$$

so that we can write (2.4) as

$$\begin{bmatrix} S_{11} & S_{12} \\ 0_{n(y)\times n(w)} & S_{22} \end{bmatrix} \mathbb{E}_t \begin{bmatrix} \tilde{\mathbf{w}}_{t+1} \\ \tilde{\mathbf{y}}_{t+1} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ 0_{n(y)\times n(w)} & T_{22} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{w}}_t \\ \tilde{\mathbf{y}}_t \end{bmatrix}. \tag{2.7}$$

[3] Heer and Maußner (2009) present an illustrative example.

[4] King and Watson (1998, 2000) present a different way to reduce a singular system of linear stochastic difference equations. The advantage of using the generalized Schur factorization, instead, is that it is implemented in the freely available LAPACK programs, and thus, easy to implement.

[5] See, e.g., Golub and van Loan (1996), Theorem 7.7.1, p. 377 who also describe the algorithm to compute the factorization of A and B. The set of all matrices of the form $B - \lambda A$, $\lambda \in \mathbb{C}$ is called a pencil. The eigenvalues of the pencil are the solutions of $|B - \lambda A| = 0$. Unitary matrices Q are complex-valued matrices whose conjugate (Hermitian) transpose equals the inverse of Q.

[6] If $s_{ii} = 0$ and $t_{ii} \neq 0$, the eigenvalue $\mu_{ii} = s_{ii}/t_{ii}$ of the pencil $|\mu A - B| = 0$ is defined and equal to zero. Therefore, we can regard λ_{ii} as 'infinite eigenvalue'.

S_{11} is a $n(w) \times n(w)$ upper triangular matrix, S_{22} is a $n(y) \times n(y)$ upper triangular matrix, and S_{12} is a $n(w) \times n(y)$ matrix. The matrices T_{11}, T_{22}, and T_{12} have corresponding dimensions.

Given these assumptions and definitions, the system

$$S_{22}\mathbb{E}_t\tilde{\mathbf{y}}_{t+1} = T_{22}\tilde{\mathbf{y}}_t$$

is unstable,[7] and to obtain a definite solution, we must set $\tilde{\mathbf{y}}_t = \mathbf{0}_{n(y)}$ for all t. Thus, from the first line of (2.7)

$$\tilde{\mathbf{w}}_{t+1} = S_{11}^{-1}T_{11}\tilde{\mathbf{w}}_t.$$

Since

$$\tilde{\mathbf{w}}_t = Z_{11}^{-1}\bar{\mathbf{w}}_t \tag{2.8}$$

from the first line of (2.6), we get

$$\bar{\mathbf{w}}_{t+1} = \underbrace{Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}}_{L^w}\bar{\mathbf{w}}_t.$$

The second line of (2.6) together with (2.8) implies

$$\bar{\mathbf{y}}_t = \underbrace{Z_{21}Z_{11}^{-1}}_{L^y}\bar{\mathbf{w}}_t.$$

The dynamics of the solved linear model can be represented by

$$\bar{\mathbf{x}}_{t+1} = L_x^x\bar{\mathbf{x}}_t + L_z^x\mathbf{z}_t, \tag{2.9a}$$

$$\bar{\mathbf{y}}_{t+1} = L_x^y\bar{\mathbf{x}}_t + L_z^y\mathbf{z}_t, \tag{2.9b}$$

$$\mathbf{z}_{t+1} = \Pi\mathbf{z}_t + \sigma\Omega\boldsymbol{\epsilon}_{t+1}. \tag{2.9c}$$

where the matrices of the linear approximation of the policy functions (2.2) are related to L^w and L^y according to

$$L^w = \begin{bmatrix} L_x^x & L_z^x \\ \mathbf{0}_{n(z)\times n(x)} & \Pi \end{bmatrix}, \quad L^y = \begin{bmatrix} L_x^y & L_z^y \end{bmatrix}.$$

[7] To see this, consider the last line of this system, which may be written

$$\mathbb{E}_t\tilde{y}_{n(y),t+1} = \lambda_{n(y),n(y)}\tilde{y}_{n(y),t}, \quad |\lambda_{n(y),n(y)}| = |(t_{n(y),n(y)}/s_{n(y),n(y)})| > 1,$$

where $s_{n(y),n(y)}$ and $t_{n(y),n(y)}$ denote the last element on the main diagonal of S_{22} and T_{22}, respectively.

2.4 Model Reduction

In this section we assume that the researcher is able to sort the equations in $\mathbf{g}(\cdot)$ so that the first $n(u)$ equations involve only period t variables. This allows us to partition $\mathbf{y}_t = [\mathbf{u}_t', \mathbf{v}_t']'$ and to write the linearized system (2.1) as:

$$C_u \bar{\mathbf{u}}_t = C_{wv} \begin{bmatrix} \bar{\mathbf{w}}_t \\ \bar{\mathbf{v}}_t \end{bmatrix}, \tag{2.10a}$$

$$D_{wv} \mathbb{E}_t \begin{bmatrix} \bar{\mathbf{w}}_{t+1} \\ \bar{\mathbf{v}}_{t+1} \end{bmatrix} + F_{wv} \begin{bmatrix} \bar{\mathbf{w}}_t \\ \bar{\mathbf{v}}_t \end{bmatrix} = \tilde{D}_u \mathbb{E}_t \bar{\mathbf{u}}_{t+1} + \tilde{F}_u \bar{\mathbf{u}}_t, \tag{2.10b}$$

where the matrices are related to the Jacobian matrix of $\mathbf{g}$ according to:

$$D\mathbf{g} = \begin{bmatrix} C_x & C_z & C_u & C_v & 0 & 0 & 0 & 0 \\ F_x & F_z & F_u & F_v & D_x & D_z & D_u & D_v \end{bmatrix},$$

$$C_{wv} = \begin{bmatrix} -C_x & -C_z - C_v \end{bmatrix}, \; D_{wv} = \begin{bmatrix} D_x & D_z & D_v \\ 0 & I_{n(z)} & 0 \end{bmatrix}, \; F_{wv} = \begin{bmatrix} F_x & F_z & F_v \\ 0 & -\Pi & 0 \end{bmatrix},$$

$$\tilde{D}_u = \begin{bmatrix} D_u \\ 0 \end{bmatrix}, \; \tilde{F}_u = \begin{bmatrix} F_u \\ 0 \end{bmatrix}.$$

Solving (2.10a) for $\mathbf{u}_t$ and using the result in (2.10b) yields:

$$\mathbb{E}_t \begin{bmatrix} \bar{\mathbf{w}}_{t+1} \\ \bar{\mathbf{v}}_{t+1} \end{bmatrix} = W \begin{bmatrix} \bar{\mathbf{w}}_t \\ \bar{\mathbf{v}}_t \end{bmatrix}, \; W = \left[D_{wv} - \tilde{D}_u C_u^{-1} C_{wv} \right]^{-1} \left[F_{wv} - \tilde{F}_u C_u^{-1} C_{wv} \right]. \tag{2.11}$$

This system can be solved along the same lines as system (2.4a). The (simple) Schur factorization of the matrix W is given by

$$S = Z^H W Z, \tag{2.12}$$

where S is an upper triangular matrix with the eigenvalues of W on the main diagonal. Assume that $n(w) = n(x) + n(z)$ eigenvalues are within and $n(v)$ eigenvalues outside the unit circle. S and Z can be chosen so that the first $n(w)$ eigenvalues appear first on the main diagonal of S. In the new variables

$$\begin{bmatrix} \bar{\mathbf{w}}_t \\ \bar{\mathbf{v}}_t \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{w}}_t \\ \tilde{\mathbf{v}}_t \end{bmatrix} \tag{2.13}$$

the transformed system reads

$$\mathbb{E}_t \begin{bmatrix} \tilde{\mathbf{w}}_{t+1} \\ \tilde{\mathbf{v}}_{t+1} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ 0_{n(y) \times n(w)} & S_{22} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{w}}_t \\ \tilde{\mathbf{v}}_t \end{bmatrix}. \tag{2.14}$$

Accordingly, the system $\mathbb{E}_t\tilde{\mathbf{v}}_{t+1} = S_{22}\tilde{\mathbf{v}}_t$ is unstable and we must set $\tilde{\mathbf{v}}_t = \mathbf{0}_{n(v)}\forall t$ so that the solution of the linear model (2.11) is

$$L^w = \begin{bmatrix} L_x^x & L_z^x \\ 0_{n(z)\times n(x)} & \Pi \end{bmatrix} = Z_{11}S_{11}Z_{11}^{-1}, \tag{2.15a}$$

$$L^v = \begin{bmatrix} L_x^v & L_z^v \end{bmatrix} = Z_{21}Z_{11}^{-1}. \tag{2.15b}$$

Using (2.15b) in (2.10a) yields

$$L^u = \begin{bmatrix} L_x^u & L_z^u \end{bmatrix} = C_u^{-1}C_{wv}\begin{bmatrix} I_{n(w)} \\ Z_{21}Z_{11}^{-1} \end{bmatrix} \tag{2.15c}$$

so that the matrices from (2.9b) are given by

$$L_x^y = \begin{bmatrix} L_x^u \\ L_x^v \end{bmatrix},\ L_z^y = \begin{bmatrix} L_z^u \\ L_z^v \end{bmatrix}. \tag{2.15d}$$

Apart from numerical imprecision, the solution (2.15) does not differ from (2.9). If we remove the first $n(u)$ equations from the system (2.4) the matrix A can be inverted and the QZ-factorization of $(B - \lambda A)$ delivers the same eigenvalues as the Schur decomposition of W. The solutions of (2.4) and (2.10) also solve (2.3). To see this, let $\bar{\mathbf{w}}_{t+1} = L^w\bar{\mathbf{w}}_t$ and $\bar{\mathbf{y}}_t = L^y\bar{\mathbf{w}}_t$ denote the solution of (2.4). This yields

$$A\begin{bmatrix} L^w \\ L^yL^w \end{bmatrix}\bar{\mathbf{w}}_t - B\begin{bmatrix} I_{n(w)} \\ L^y \end{bmatrix}\bar{\mathbf{w}}_t = \mathbf{0}_{n(w)+n(y)}.$$

The coefficients of L^w and L^y, thus, must satisfy the system of equations

$$A\begin{bmatrix} L^w \\ L^yL^w \end{bmatrix} - B\begin{bmatrix} I_{n(w)} \\ L^y \end{bmatrix} = \mathbf{0}_{(n(w)+n(y))\times n(w)} \tag{2.16}$$

which is just the matrix version of (2.3). Note that (2.16) provides a simple way to check the accuracy of solutions based on matrix methods against the solution that would solve the non-linear system (2.3). Given the matrices A and B and the policy functions L^w and L^y the entries of the $(n(w)+n(y))\times n(w)$ matrix on the left-hand-side of (2.16) should not exceed a given tolerance, 10^{-6}, say, in absolute value.

2.5 Implementation

The linear algebra package (LAPACK) provides several routines to compute both the QZ-factorization of a matrix pencil and the Schur decomposition of non-symmetric matrices.[8] The Fortran program `Solab.f90` by Paul Klein, which is also used by

[8]The routines are written in Fortran. C interfaces to these routines also exist.

DYNARE, employs the LAPACK routine `ZGGES` to factor the pencil $(B - \lambda A)$. We use the LAPACK routines `ZGGESX` and `ZGGESXV`. Both `ZGGES` and `ZGGESX` compute the matrices S, T, Q, and Z and clusters the eigenvalues in two blocks. The eigenvalues are given by $\lambda_i = \alpha_i/\beta_i$, where α_i and β_i are equal to the diagonal elements of S and T, respectively. Klein's procedure requires to cluster the eigenvalues according to the criterium $|\alpha_i| > |\beta_i|$ (or $|\lambda_i| > 1$) so that the eigenvalues of $S_{11}^{-1}T_{11}$ are within the unit circle.[9] `ZGGESX` additionally computes average reciprocal condition numbers κ for the eigenvalue clusters. An approximate error bound for each cluster is given by[10]

$$\chi(\bar{\lambda}, \bar{\lambda}') = \frac{|\bar{\alpha}\bar{\beta}' - \bar{\beta}\bar{\alpha}'|}{\sqrt{|\bar{\alpha}|^2| + |\bar{\beta}|^2}\sqrt{|\bar{\alpha}'|^2| + \bar{\beta}'|^2}} \leq \frac{\epsilon\sqrt{||A||_1^2 + ||B||_1^2}}{\kappa}, \tag{2.17}$$

where χ is the chordal distance between the average eigenvalue in the selected cluster $\bar{\lambda} = \sum_{i=1}^{n(w)} \lambda_i/(n(w))$ and the average true eigenvalue $\bar{\lambda}'$ of the pencil. ϵ refers to the machine epsilon, $||A||_1$ and $||B||_1$ are the one-norms of A and B, respectively. They are returned from `ZGGESXV`. The routines `ZGGES` and `ZGGESX` balance A and B by scaling the rows and columns to reduce computational errors. Therefore, the error bounds refer to the scaled matrices and not to A and B as passed to `ZGGES` and `ZGGESX`. As will become apparent later, an additional scaling may be necessary to reduce computational errors.

The LAPACK routine `ZGEESX` performs the Schur decomposition (2.12), clusters eigenvalues so that the eigenvalues of S_{11} are within the unit circle, and provides reciprocal condition numbers κ_i, $i = 1, 2$, for the average eigenvalues of S_{11} and S_{22}. An approximate error bound for the distance between the true average eigenvalue $\bar{\lambda}'$ of S_{11}, and the computed average $\bar{\lambda}$ is given by

$$[|\bar{\lambda} - \bar{\lambda}'| \leq \frac{\epsilon||W||_1}{\kappa_1}. \tag{2.18}$$

As in the case of the generalized eigenvalue problem, `ZGEESX` scales the rows and columns of the matrix W to reduce the computational errors. Therefore, the one-norm of W returned by `ZGEEVX` is the norm of the scaled matrix.

3 An Example

We consider a real business cycle model taken from Heer and Maußner (2013) that features habits in consumption and hours as well as adjustment costs of capital. The

[9] The procedure of Heer and Maußner (2009) selects the eigenvalues according to $|\mu_i| < 1$, $\mu_i = \alpha_i/\beta_i$.

[10] See the LAPACK Users Guide on `http://www.netlib.org/lapack/lug/node1.html`.

model introduces endogenous labor supply in the model of Jermann (1998) who studied the equity premium implications of a production economy.

3.1 The Model

Households. Households enter the current period t with a given amount of firm shares S_t and real bonds B_t. Bonds have a maturity of one period and can be purchased at the current price v_t^b and pay one unit of consumption in period $t+1$. The real share price is v_t^e and real dividend payments per share are d_t. Firms pay the real wage w_t per unit of working hours N_t. Thus,

$$v_t^e(S_{t+1} - S_t) + v_t^b B_{t+1} \leq w_t N_t + d_t S_t + B_t - C_t \tag{3.1}$$

is the household's budget constraint, where C_t denotes consumption. The household chooses contingency plans for consumption C_t, hours N_t, and next-period stocks S_{t+1} that maximize

$$U_t = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \frac{(C_{t+s} - \chi^C C_{t+s-1})^{1-\eta} - 1}{1-\eta} - \nu_0 \frac{(N_{t+s} - \chi^N N_{t+s-1})^{1+\nu_1} - 1}{1+\nu_1} \tag{3.2}$$

subject to (3.1). The parameters χ^C and χ^N determine the degree of habits in consumption and labor supply. We treat both habits as exogenous, i.e., C_{t+s-1} and N_{t+s-1} refer to average consumption and labor supply of the previous period. The first-order conditions for this problem and any further mathematical details of this model are presented in the Appendix.

Firms. The representative firm uses labor N_t and capital K_t to produce output Y_t according to the production function

$$Y_t = Z_t N_t^{1-\alpha} K_t^{\alpha}, \quad \alpha \in (0,1). \tag{3.3}$$

The level of total factor productivity Z_t is governed by the AR(1)-Process

$$\ln Z_t = \rho \ln Z_{t-1} + \sigma \epsilon_t, \quad \epsilon_t \sim \mathcal{N}(0,1). \tag{3.4}$$

The firm finances part of its investment I_t from retained earnings RE_t and issues new shares to cover the remaining part:

$$I_t = v_t(S_{t+1} - S_t) + RE_t. \tag{3.5}$$

It distributes the excess of its profits over retained earnings to the household sector:

$$d_t S_t = Y_t - w_t N_t - RE_t. \tag{3.6}$$

Investment increases the firm's future stock of capital according to:

$$K_{t+1} = \Phi(I_t/K_t)K_t + (1-\delta)K_t, \quad \delta \in [0,1], \tag{3.7}$$

where we parameterize the function Φ as

$$\Phi(I_t/K_t) := \frac{a_1}{1-\zeta}\left(\frac{I_t}{K_t}\right)^{1-\zeta} + a_2, \quad \zeta > 0, \tag{3.8}$$

and determine a_1 and a_2 from $\Phi'(\delta) = 1$ and $\Phi(\delta) = \delta$ so that adjustment costs are absent in the deterministic stationary equilibrium.

The firm maximizes its beginning-of-period value

$$V_t = \mathbb{E}_t \sum_{s=0}^{\infty} \varrho_{t+s}(Y_{t+s} - w_{t+s}N_{t+s} - I_{t+s}) \tag{3.9}$$

subject to (3.3) and (3.7). The variable

$$\varrho_{t+s} = \beta^s \frac{\Lambda_{t+s}}{\Lambda_t}$$

is the household's stochastic discount factor for period $t+s$ returns and Λ_t equals the marginal utility of consumption

$$\Lambda_t = (C_t - \chi^C C_{t-1})^{-\eta}. \tag{3.10}$$

The respective first-order conditions can be found in the Appendix.

Calibration. We calibrate the model with respect to the US economy. Table 3.1 displays our choice of parameters. The standard parameter values for the production side, α, ρ, and σ are taken from Hansen (1985), as well as the value of the discount factor β. The habit parameter χ^C, η, and the parameters of the capital accumulation equation (3.7) are taken from Jermann (1998). The parameter $\nu_1 = 2.5$ is from de Paoli, Scott, and Weeken (2010). As these authors, we assume $\chi^N = \chi^C$,[11] and choose ν_0 so that in the stationary equilibrium N equals 0.33.

[11] Alternatively, following Heer and Maußner (2013), we could have chosen the unobserved parameters so that the model replicates certain empirical facts. Yet, since we use the model just as an example, the precise calibration does not matter.

Table 3.1
Parameter Choice

Preferences	β=0.99	η=5	ν_1=2.5	$N\in\{0.13, 0.33\}$
	χ^C=0.82	χ^N=0.82		
Production	α=0.36	ρ=0.95	σ=0.00712	
Capital Accumulation	δ=0.025	ζ=1/0.23		

3.2 Accuracy of the Solution

Solutions. In this Section we consider the linear part of the model's approximate solution. We compute four different linear solutions for the levels of the variables:

#1: the solution of the non-linear system via a non-linear equations solver, with termination criterium $\max_k |f_k(L^w, L^y)| < 10^{-7}$, where f_k, $k = 1, \ldots, (n(w) + n(y)) \times n(w)$ denotes the kth equation of (2.3),

#2: the solution via the QZ-factorization of the matrix pencil $(B - \lambda A)$,

#3: the solution via the QZ-factorization oft the matrix pencil $(A - \mu B)$,

#4: the solution of the reduced system, where we partition

$$\mathbf{y}_t = [\mathbf{u}_t', \mathbf{v}_t']' = [Y_t, C_t, I_t, N_t, w_t, q_t, \Lambda_t]'$$

so that $\mathbf{v_t} \equiv \Lambda_t$.[12].

As shown in Section 2, if we ignore different degrees of numerical precision, all four solutions should deliver the same policy functions.

Here we report the solutions from our Fortran program. On our home page we also provide a Maple program and a DYNARE script.[13] While the Fortran program employs a central difference approximation of the Jacobian matrix of $\mathbf{g}$, the Maple program as well as DYNARE use symbolic algebra to derive the analytic formulas for the elements of the Jacobian and evaluates these at the stationary solution. The results reported

[12] Λ_t denote the Lagrange multiplier of budget constraint (3.1).

[13] See `http://www.wiwi.uni-augsburg.de/vwl/maussner/lehrstuhl/maussner_en.html`.

below are robust with respect to the computation of the Jacobian. The policy functions from DYNARE reproduce solution #2.

With these solutions we also simulate the model and report second moments for the percentage deviations of the model's variables from their respective steady state values.[14] We do the latter for three reasons. First, researchers usually do not report the coefficients of the policy functions but present statistics that summarize the empirical implications of their model. For this reason it is important to see whether or not simulated measures of the business cycle bear traces of the numerical differences in the coefficients of the policy functions. Second, as is well known from the benchmark business cycle model, different degrees of numerical accuracy rarely surface in differences of second moments from simulated time series.[15] Third, as we demonstrate in the Appendix, if we would solve and simulate the log-linearized version of the model, the respective second moments would not depend on the value of N, the stationary fraction of hours supplied by the representative agent. Therefore, any discrepancies we will observe indicate serious numerical differences between the four different solutions.

Policy Functions. Tables A.2 and A.3 in the Appendix report the coefficients of the policy functions (i.e., the linear approximate solution of the model) for the case of $N = 0.13$ and $N = 1/3$. To save on space, here we focus on the relative difference between solutions #2 through #4 and solution #1.

In the case of $N = 0.33$ the coefficients are virtually identical: the maximum relative difference between the coefficients is less than 0.005 percent and relates to the coefficient of Tobin's q with respect to the capital stock as computed by solution # 3. This changes considerably, if we use $N = 0.13$, a value used by Heer and Maußner (2013) for the German economy.

Table 3.2 presents the results for this case. There are virtually no differences between the solution of the reduced system and the non-linear solution. In absolute terms the maximum relative distance between the solution based on the QZ-factorization $(B-\lambda A)$ and the non-linear solutions is 65 percent for the coefficient of q_t with respect to capital K_t: the non-linear solution provides a coefficient of about -0.024 while solution # 2 yields about -0.008. For the same coefficient, the QZ-factorization of $(B - \mu A)$

[14] We report the steady state solution of the deterministic counterpart of the model in Table A.1 in the Appendix.

[15] See Aruoba et al. (2006) and Heer and Maußner (2008).

Table 3.2

Policy Functions for $N = 0.13$, Relative Discrepancy

Dependent Variables	Independent Variables K_t	C_{t-1}	N_{t-1}	$\ln Z_t$
		$(B - \lambda A)$ versus non-linear system		
K_{t+1}	0.000446	0.005025	0.004494	0.017461
Y_t	0.009649	−0.003991	0.001928	0.007353
C_t	−0.036036	0.001049	−0.003646	−0.013363
I_t	0.018323	0.005025	0.004494	0.017461
N_t	−0.045245	−0.003991	0.001928	−0.015204
w_t	−0.004071	−0.003991	0.001928	−0.002356
q_t	−0.650130	0.005025	0.004494	0.017461
Λ_t	−0.036036	−0.003991	−0.003646	−0.013363
		$(A - \mu B)$ versus non-linear system		
K_{t+1}	−0.008139	−1.019427	−1.000000	−1.000000
Y_t	0.027064	−0.018830	0.009491	0.023591
C_t	−0.101083	0.004949	−0.017945	−0.042873
I_t	0.043836	0.003276	0.002099	0.035338
N_t	−0.126914	−0.018830	0.009491	−0.048780
w_t	−0.011419	−0.018830	0.009491	−0.007558
q_t	−1.566015	0.004084	0.002891	0.036156
Λ_t	−0.101083	−0.018830	−0.017945	−0.042873
		Reduced system versus non-linear system		
K_{t+1}	0.000000	0.000000	0.000000	0.000000
Y_t	0.000000	0.000000	0.000000	0.000000
C_t	0.000000	0.000000	0.000000	0.000000
I_t	0.000000	0.000000	0.000000	0.000000
N_t	0.000000	0.000000	0.000000	0.000000
w_t	0.000000	0.000000	0.000000	0.000000
q_t	0.000001	0.000000	0.000000	0.000000
Λ_t	0.000000	0.000000	0.000000	0.000000

Notes: The entries represent relative differences between the coefficients of the policy functions of the variables in the leftmost column. Λ_t is the Lagrange multiplier of the household's budget constraint.

(solution # 3) even delivers a positive value of about 0.014 which gives raise to a relative difference of over 150 percent. While the non-linear solution gives positive coefficients in the policy function for the future capital stock K_{t+1} for N_{t-1} and $\ln Z_t$ of 0.85 and 0.22, respectively, solution #3 yields zero coefficients, which explain the 100 percent deviation in the entries for K_{t+1} in Table 3.2.

Second Moments. Table 3.3 presents results from six different simulations of the model. The moments in the first and second panel rest on the solution # 2 and #3, respectively, while the moments in the third panel are from simulations that use the policy function obtained from the non-linear solution #1. The single difference between the panels labeled $N = 0.33$ and $N = 0.13$ are two different values for the stationary level of hours N. The second moments refer to percentage deviations of a variable from its stationary solution. They were computed as averages over 500 simulations. Each individual time series has 200 observations. We show in the Appendix that the coefficient matrices of the log-linearized system do not depend on N so that the simulations should yield identical second moments, given that the same sequence of random numbers is used. Obviously, this is true for the non-linear solution (and also, but not shown in the Table, for solution #4). However, the second moments obtained from solutions #2 and #3 reveal many obvious and large differences, both in the standard deviations and in the cross- as well as autocorrelations of the variables displayed.

Euler Equation Residuals. Euler equation residuals are frequently employed to investigate the degree of numerical accuracy. In Table 3.4 we report the maximum residuals of the Euler equation for capital:[16]

$$\Lambda_t = \beta \mathbb{E}_t \frac{\Lambda_{t+1}[\alpha(Y_{t+1}/K_{t+1}) - (I_{t+1}/K_{t+1}) + q_{t+1}(1 - \delta + \Phi(I_{t+1}/K_{t+1}))]}{q_t}. \tag{3.11}$$

We compute the residuals on a grid $\mathscr{G}$ defined on the four dimensional cube around the stationary solution of the variables K, C, N, and $Z = 1$ given by:[17]

$$[0.93K, 1.07K] \times [0.93C, 1.07C] \times [0.94N, 1.04N] \times [0.95, 1.05].$$

Each of the subintervals is discretized in 50 points. For each $(K_t, C_{t-1}, N_{t-1}, Z_t) \in \mathscr{G}$ we evaluate the right-hand side of equation (3.11) by Gauss-Hermite integration with

[16] This equation is derived in the Appendix.

[17] On a larger cube the policy functions occasionally return negative values so that a meaningful comparison is not possible.

Table 3.3

Second Moments

Variable	s_x	s_x/s_Y	r_{xY}	r_x	s_x	s_x/s_Y	r_{xY}	r_x
	$N = 0.33$				$N = 0.13$			
				$(A - \mu B)$				
Output	1.19	1.00	1.00	0.90	0.82	1.00	1.00	0.79
Consumption	1.09	0.91	0.97	0.97	0.34	0.41	1.00	0.79
Investment	1.77	1.48	0.90	0.67	2.17	2.65	1.00	0.79
Hours	1.37	1.15	−0.97	0.96	1.92	2.34	−0.84	0.98
Real Wage	2.55	2.13	0.99	0.94	2.65	3.23	0.92	0.94
Tobin's q	7.09	5.93	0.79	0.61	9.46	11.54	1.00	0.79
				$(B - \lambda A)$				
Output	1.19	1.00	1.00	0.90	0.82	1.00	1.00	0.75
Consumption	1.09	0.91	0.97	0.97	0.70	0.86	0.94	0.95
Investment	1.77	1.48	0.90	0.67	1.44	1.75	0.89	0.31
Hours	1.37	1.15	−0.97	0.96	1.98	2.42	−0.88	0.96
Real Wage	2.55	2.13	0.99	0.94	2.73	3.34	0.94	0.94
Tobin's q	7.09	5.93	0.79	0.61	5.97	7.29	0.80	0.24
				Non-linear solution				
Output	1.19	1.00	1.00	0.90	1.19	1.00	1.00	0.90
Consumption	1.09	0.91	0.97	0.97	1.09	0.91	0.97	0.97
Investment	1.77	1.48	0.90	0.67	1.77	1.48	0.90	0.67
Hours	1.37	1.15	−0.97	0.96	1.37	1.15	−0.97	0.96
Real Wage	2.55	2.13	0.99	0.94	2.55	2.13	0.99	0.94
Tobin's q	7.09	5.93	0.79	0.61	7.09	5.93	0.79	0.61

Notes: s_x:=Standard deviation of percentage deviations of variable x from its stationary solution. x stands for any of the variables from column 1. Results are from 500 replications with 200 observations each. s_x/s_Y:=Standard deviation of variable x relative to standard deviation of output Y. r_{xY}:=Cross-correlation of variable x with output y, r_x:=First order autocorrelation of variable x.

four points and use the respective value of Λ_t to compute the amount of consumption $\hat{C}_t$ that would deliver a zero residual from equation (3.10). The residual is defined as $(\hat{C}_t/C_t) - 1$, where C_t is the amount of consumption determined from the policy function (see Judd and Guu (1997)).

The Euler equation residuals confirm the results from the simulations, albeit not as

Table 3.4
Euler Equation Residuals

Solution	$N = 1/3$	$N = 0.13$
#1	0.04519	0.04519
#2	0.04519	0.04711
#3	0.04519	0.04860
#4	0.04519	0.04519

Notes: The left-most column indicates the number of the solution as defined in the body of the paper.

impressive.[18] Solution #3 obtained from the QZ-decomposition of $(A - \mu B)$ is the least accurate one followed by solution #2. The consumption equivalent of the former (latter) is about 4.9 percent (4.7) as compared to 4.5 percent for the more accurate non-linear solution #1 and the solution of the reduced system. For the case $N = 1/3$ all four solutions deliver the same Euler residual.

3.3 Source of the Problem and Remedies

Unbalanced Matrices. The odd results reported in the previous subsection origin in two equations, the first-order condition for consumption and for labor supply:

$$\Lambda_t = (C_t - \chi^C C_{t-1})^{-\eta}, \tag{3.12a}$$

$$\Lambda_t w_t = \nu_0 (N_t - \chi^N N_{t-1})^{\nu_1}. \tag{3.12b}$$

The steady-state value of consumption is small and increases with the stationary value of working hours N. Therefore, a strong habit (χ^C close to one) and a large coefficient of relative risk aversion η imply a huge value of Λ, the multiplier of the budget constraint (3.1). This gives rise to very large coefficients in the Jacobian matrix of $\mathbf{g}$, and, accordingly, in the matrix B of (2.4a) and the matrix W of (2.11). Yet, due to the reduction of the model, W is less unbalanced than A.

[18]Note however that the Euler equation residuals inform about the error of an approximate solution while Table 3.2 informs about different degrees of accuracy of the linear solution. It may still happen that numerically inaccurate coefficients in the linearized policy functions result into a better approximation. However this is not the case here.

The different degrees of accuracy for $N = 0.13$ and $N = 1/3$ can also be seen from the approximate error bounds (2.17) and (2.18). Due to the scaling performed by `ZGGESX` they do not differ between the two different versions of the QZ-factorization. Yet, for the cluster of eigenvalues within the unit circle the approximate error bound for the case $N = 0.13$ is about 1.9×10^2 larger in magnitude than for the case of $N = 0.33$. The error bound for the Schur decomposition is about 6.2×10^3 larger.

Transformation and Non-linear solution. One way to overcome this problem is a reformulation of (3.12):

$$1 = \frac{(C_t - \chi^C C_{t-1})^{-\eta}}{\Lambda_t},$$
$$1 = \nu_0 \frac{(N_t - \chi^N N_{t-1})^{\nu_1}}{w_t \Lambda_t}.$$

Indeed, we find negligible differences between the policy function of the different solutions after this change. However, it may not always be obvious how to reformulate a model's equations or to transform its variables. Of course, one might be tempted to always resort to the non-linear solution. Yet, this requires additional programming and unexperienced users of programming software might hesitate to take this step.

Balancing. However, there is third possible solution: a previous balancing of the matrix pencil. While the scaling performed by `ZGGESX` aims to make the elements of the scaled matrices A and B as close as possible to unity (see Ward (1981)), the scaling proposed as by Lemonnier and van Dooren (2006) tries to make a matrix pencil as similar as possible to a pencil with orthogonal left and right eigenvectors.[19] The algorithm of Lemonnier and Dooren (2006) computes two diagonal matrices C and D so that $A' = C^{-1}AD$ and $B' = C^{-1}BD$ represent the scaled pencil. We let `ZGGESX` factor this pencil, solved for the policy functions of the transformed problem, and transformed these back to those of the original problem. In this way we were able to reduce the maximum relative error between the QZ-factorizations and the non-linear solution to less than $.6 \times 10^{-12}$. The success of this balancing scheme to reduce computational errors can also be seen from the error bound (2.17). In the case of $N = 0.13$ the error for the eigenvalues within the first cluster drops from 0.22×10^{-7} to 0.12×10^{-12} for solution #2 and from 0.20×10^{-7} to 0.49×10^{-13} for solution # 3. For both solutions the Euler equation residual drops to 4.5 percent.

[19] We are grateful to the referee who pointed us to this article.

4 Conclusion

The availability of easy to use toolkits for solving dynamic stochastic general equilibrium (DSGE) models has enhanced the widespread application of these models in macroeconomic research. The researcher supplies the equations of his model to programs as, e.g., Dynare, which solve and simulate the model.

We demonstrate by means of an example that an uninformed use of DSGE solution software may produce strange results. Researchers should be aware of those effects and take appropriate measures.

We consider a model that has been employed in studies of the equity premium puzzle. Due to matrices with very large and very small coefficients the policy functions obtained from different ways to solve the model differ widely. The differences in policy functions give raise to differences in the second moments of model simulated time series so that the researcher may be misled with respect to the dynamic properties of the model. A very effective and particularly simple way to deal with this problem is the previous balancing of the matrices of the linearized system. The scheme which we employ differs from the scaling routinely undertaken by the procedures from the linear algebra package. Transformations of the model's variables and equations as well as solving a system of non-linear equations are alternative, but less straight forward strategies to handle the problem.

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Appendix (not for publication)

In this Appendix, we provide the formal details of our example model, present tables with the policy functions from six different solutions, and compute several measures of the accuracy of the linear solutions.

A.1 The Model

Equilibrium Conditions. The first-order conditions of maximizing (3.2) subject to (3.1) and given initial values of S_t and B_t are:

$$\Lambda_t = (C_t - \chi^C C_{t-1})^{-\eta}, \tag{A.1a}$$

$$\Lambda_t w_t = \nu_0 (N_t - \chi^N N_{t-1})^{\nu_1}, \tag{A.1b}$$

$$v_t^e = \beta \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t}(d_{t+1} + v_{t+1}^e), \tag{A.1c}$$

$$v_t^b = \beta \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t}, \tag{A.1d}$$

where Λ_t is the Lagrange multiplier of the budget constraint (3.1).

The first-order conditions of the firm's problem – maximizing (3.9) subject to (3.3) and (3.7) and a given initial stock of capital K_t – are

$$w_t = (1-\alpha) Z_t N_t^{-\alpha} K_t^{\alpha}, \tag{A.2a}$$

$$q_t = \frac{1}{\Phi'(I_t/K_t)}, \tag{A.2b}$$

$$q_t \varrho_t = \mathbb{E}_t \varrho_{t+1} \Big\{ \alpha Z_{t+1} N_{t+1}^{1-\alpha} K_{t+1}^{\alpha-1} - (I_{t+1}/K_{t+1}) + q_{t+1} \big[\Phi(I_{t+1}/K_{t+1}) + 1 - \delta \big] \Big\}. \tag{A.2c}$$

In equilibrium all markets clear. We assume that bonds are in zero supply, $B_t = 0 \forall t$, and make no assumption with respect the dividend policy. Accordingly, we disregard d_t and v_t^e. Using equations (3.5) and (3.6) the household's budget constraint (3.1) reduces to the economy's resource restriction $Y_t = C_t + I_t$. Equilibrium in the market for shares requires

$$\varrho_{t+s} = \beta^s \frac{\Lambda_{t+s}}{\Lambda_t}.$$

Let $\mathbf{x}_t = [K_t, C_{t-1}, N_{t-1}]'$, $\mathbf{y}_t := [Y_t, C_t, I_t, N_t, w_t, q_t, \Lambda_t]'$, $\mathbf{z}_t := \ln Z_t$. Then, the system (2.1) is given by:

$$\Lambda_t = (C_t - \chi^C C_{t-1})^{-\eta}, \tag{A.3a}$$

$$\Lambda_t w_t = \nu_0 (N_t - \chi^N N_{t-1})^{\nu_1}, \tag{A.3b}$$

$$w_t = (1-\alpha) Z_t N_t^{-\alpha} K_t^{\alpha}, \tag{A.3c}$$

$$q_t = \frac{1}{\Phi'(I_t/K_t)}, \tag{A.3d}$$

$$Y_t = Z_t N_t^{1-\alpha} K_t^{\alpha}, \tag{A.3e}$$

$$Y_t = C_t + I_t, \tag{A.3f}$$

$$q_t = \beta \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda t} \Big\{ \alpha Z_{t+1} N_{t+1}^{1-\alpha} K_{t+1}^{\alpha-1} - (I_{t+1}/K_{t+1}) + q_{t+1} \big[\Phi(I_{t+1}/K_{t+1}) + 1 - \delta \big] \Big\} \tag{A.3g}$$

$$K_{t+1} = \Phi(I_t/K_t) K_t + (1-\delta) K_t. \tag{A.3h}$$

Note that equations (A.3a)-(A.3f) involve only variables dated at t (using the definition of $\mathbf{x}_t$ above). Therefore, the matrix A of the linearized model will be singular.

Deterministic Stationary Equilibrium. Assume $\sigma = 0$ so that $\ln Z_t$ equals its unconditional expectation 0 for all t (and, hence, $Z = 1$). In this case, we can ignore the expectations operator $\mathbb{E}_t$. Stationarity implies $x_{t+1} = x_t = x$ for any variable in our model. As usual, we specify Φ so that adjustment costs play no role in the stationary equilibrium, i.e., $\Phi(I/K)K = \delta K$ and $q = \Phi'(\delta) = 1$. This requires that we choose

$$a_1 = \delta^{\zeta}, \tag{A.4a}$$

$$a_2 = \frac{-\zeta\delta}{1-\zeta}. \tag{A.4b}$$

These assumptions imply via equation (A.3g) the stationary solution for the output-capital-ratio:

$$\frac{Y}{K} = \frac{1-\beta(1-\delta)}{\alpha\beta}. \tag{A.5a}$$

Using the production function, we can solve for the capital-labor ratio and for labor productivity:

$$\frac{K}{N} = \left(\frac{Y}{K}\right)^{\frac{1}{\alpha-1}}, \tag{A.5b}$$

$$\frac{Y}{N} = \left(\frac{Y}{K}\right)^{\frac{\alpha}{\alpha-1}}. \tag{A.5c}$$

Given these solutions equations (A.3a)-(A.3c) and (A.3e) can be reduced to an equation in N:

$$(1-\alpha)\frac{Y}{N} = \nu_0 (1-\chi^N)^{\nu_1} (1-\chi^C)^{\eta} \left(\frac{Y}{N} - \delta\frac{K}{N}\right)^{\eta} N^{\nu_1+\eta}. \tag{A.5d}$$

We solve this equation for the parameter ν_0 given our calibration target for N. The levels of the stock of capital K, output Y, consumption C, and investment I can be computed from (A.5a)-(A.5c). In the final step, equation (A.3a) delivers the stationary level of the Lagrange multiplier Λ.

Table A.1 presents the values of the model's variables for both values of N. The values of the parameter ν_0 is determined from equation (A.3b) so as to imply the given N.

Table A.1

Stationary Solution

Variable	$N = 1/3$	$N = 0.13$
Y	$0.123469E+01$	$0.481528E+00$
C	$0.918109E+00$	$0.358063E+00$
I	$0.316577E+00$	$0.123465E+00$
K	$0.126631E+02$	$0.493860E+01$
w	$0.237060E+01$	$0.237060E+01$
q	$0.100000E+01$	$0.100000E+01$
Λ	$0.811271E+04$	$0.899172E+06$
ν_0	$0.218095E+08$	$0.254484E+11$
a_1	$0.108273E-06$	$0.108273E-06$
a_2	$0.324675E-01$	$0.324675E-01$

The smaller value of $N = 0.13$ implies smaller values of output Y, consumption C, investment I, and the stock of capital K. Accordingly, the value of Λ computed from (A.3a) is much larger. The values of the parameters of the capital adjustment function Φ defined in equation (3.8) are independent of the value of N (see equations (A.4)).

The Loglinear System. Let $\hat{x}_t \equiv (x_t - x)/x \simeq \ln(x_t/x)$ denote the percentage deviation of x_t from its stationary solution x. In terms of this variables the linearized system (A.3b) reads:

$$\hat{\Lambda}_t = \frac{-\eta}{1-\chi^C}\hat{C}_t + \frac{\eta\chi^C}{1-\chi^C}\hat{C}_{t-1}, \tag{A.6a}$$

$$\hat{\Lambda}_t = -\hat{w}_t + \frac{\nu_1}{1-\chi^N}\hat{N}_t - \frac{\nu_1\chi^N}{1-\chi^N}\hat{N}_{t-1}, \tag{A.6b}$$

$$\hat{w}_t = \hat{Z}_t - \alpha\hat{N}_t + \alpha\hat{K}_t, \tag{A.6c}$$

$$\hat{q}_t = \zeta\hat{I}_t - \zeta\hat{K}_t, \tag{A.6d}$$

$$\hat{Y}_t = \hat{Z}_t + (1-\alpha)\hat{N}_t + \alpha\hat{K}_t, \tag{A.6e}$$

$$\hat{Y}_t = \frac{C}{Y}\hat{C}_t + \frac{I}{Y}\hat{I}_t, \tag{A.6f}$$

$$\hat{q}_t = \mathbb{E}_t\hat{\Lambda}_{t+1} - \hat{\Lambda}_t + \beta\mathbb{E}_t\hat{q}_{t+1} + \alpha\beta\frac{Y}{K}\mathbb{E}_t\left[\hat{Z}_{t+1} + (1-\alpha)\hat{N}_{t+1} + (\alpha-1)\hat{K}_{t+1}\right], \tag{A.6g}$$

$$\hat{K}_{t+1} = (1-\delta)\hat{K}_t + \delta\hat{I}_t. \tag{A.6h}$$

None of the coefficients in this system depends on the level of hours N: According to (A.5a), Y/K is a function of the parameters α, β, and δ. The variable I/K is equal to $\delta/(Y/K)$ and C/Y is equal to $C/Y = (C/K)/(Y/K) = (1-\delta/(Y/K))$.

A.2 Policy Functions

Table A.2 and A.3 display the coefficients of the policy functions for $N = 0.13$ and $N = 0.334$, respectively. The rows labeled a through f refer to six different solutions of the model as explained in the Notes to Table A.2.

Table A.2: Policy Functions: $N = 0.13$

Dependent Variables		Independent Variables			
		K_t	C_{t-1}	N_{t-1}	$\ln Z_t$
K_{t+1}	a	0.999760	−0.364790	0.852468	0.221933
	b	0.999315	−0.362966	0.848655	0.218124
	c	0.991181	0.007051	0.000000	0.000000
	d	0.999315	−0.362966	0.848655	0.218124
	e	0.999315	−0.362966	0.848655	0.218124
	f	0.999315	−0.362966	0.848655	0.218124
Y_t	a	0.029211	0.285220	1.241699	0.326950
	b	0.028931	0.286363	1.239309	0.324564
	c	0.029714	0.280971	1.251072	0.332221
	d	0.028931	0.286363	1.239309	0.324564
	e	0.028931	0.286363	1.239309	0.324564
	f	0.028931	0.286363	1.239309	0.324564

Table A.2 continued

Dependent Variables		Independent Variables			
		K_t	C_{t-1}	N_{t-1}	$\ln Z_t$
C_t	a	0.004450	0.650011	0.389230	0.105017
	b	0.004617	0.649329	0.390655	0.106440
	c	0.004150	0.652543	0.383644	0.101876
	d	0.004617	0.649329	0.390655	0.106440
	e	0.004617	0.649329	0.390655	0.106440
	f	0.004617	0.649329	0.390655	0.106440
I_t	a	0.024760	−0.364790	0.852468	0.221933
	b	0.024315	−0.362966	0.848655	0.218124
	c	0.025381	−0.364155	0.850436	0.225832
	d	0.024315	−0.362966	0.848655	0.218124
	e	0.024315	−0.362966	0.848655	0.218124
	f	0.024315	−0.362966	0.848655	0.218124
N_t	a	−0.002485	0.120316	0.523791	−0.065206
	b	−0.002603	0.120798	0.522783	−0.066213
	c	−0.002272	0.118523	0.527745	−0.062983
	d	−0.002603	0.120798	0.522783	−0.066213
	e	−0.002603	0.120798	0.522783	−0.066213
	f	−0.002603	0.120798	0.522783	−0.066213
w_t	a	0.189117	−0.789841	−3.438550	2.798658
	b	0.189890	−0.793006	−3.431933	2.805267
	c	0.187722	−0.778074	−3.464506	2.784063
	d	0.189890	−0.793006	−3.431933	2.805267
	e	0.189890	−0.793006	−3.431933	2.805267
	f	0.189890	−0.793006	−3.431933	2.805267
q_t	a	−0.008443	−12.846094	30.019700	7.815374
	b	−0.024132	−12.781868	29.885393	7.681249
	c	0.013659	−12.834069	29.971782	7.958970
	d	−0.024132	−12.781868	29.885393	7.681249
	e	−0.024132	−12.781868	29.885393	7.681249
	f	−0.024132	−12.781868	29.885393	7.681249

Table A.2 continued

Dependent Variables		Independent Variables			
		K_t	C_{t-1}	N_{t-1}	$\ln Z_t$
Λ_t	a	−310435.585054	11857778.135886	−27151143.432483	−7325585.987312
	b	−322040.799510	11905287.194025	−27250492.309927	−7424799.941246
	c	−289487.870911	11681115.175497	−26761474.970035	−7106477.042567
	d	−322040.799510	11905287.194025	−27250492.309927	−7424799.941246
	e	−322040.809449	11905287.238673	−27250492.350733	−7424800.029568
	f	−322040.799510	11905287.194025	−27250492.309927	−7424799.941246

Notes: The entries represent the coefficients of the policy functions for the variables listed in the first column. The labels a-f refer to different solutions:

a: factorization of the pencil $(B - \lambda A)$, with A and B unbalanced,

b: factorization of the pencil $(B - \lambda A)$, with A and B balanced,

c: factorization of the pencil $(A - \mu B)$, with A and B unbalanced,

d: factorization of the pencil $(A - \mu B)$, with A and B balanced,

e: solution of the reduced system,

f: solution of the non-linear system.

Table A.3: Policy Functions: $N = 1/3$

Dependent Variables		Independent Variables			
		K_t	C_{t-1}	N_{t-1}	$\ln Z_t$
K_{t+1}	a	0.999315	−0.362966	0.848655	0.559293
	b	0.999315	−0.362966	0.848655	0.559293
	c	0.999315	−0.362968	0.848658	0.559294
	d	0.999315	−0.362966	0.848655	0.559293
	e	0.999315	−0.362966	0.848655	0.559293
	f	0.999315	−0.362966	0.848655	0.559293
Y_t	a	0.028931	0.286363	1.239309	0.832215
	b	0.028931	0.286363	1.239309	0.832215
	c	0.028931	0.286363	1.239309	0.832214
	d	0.028931	0.286363	1.239309	0.832215
	e	0.028931	0.286363	1.239309	0.832215
	f	0.028931	0.286363	1.239309	0.832215

Table A.3 continued

Dependent Variables		Independent Variables			
		K_t	C_{t-1}	N_{t-1}	$\ln Z_t$
C_t	a	0.004617	0.649329	0.390655	0.272922
	b	0.004617	0.649329	0.390655	0.272922
	c	0.004617	0.649329	0.390655	0.272922
	d	0.004617	0.649329	0.390655	0.272922
	e	0.004617	0.649329	0.390655	0.272922
	f	0.004617	0.649329	0.390655	0.272922
I_t	a	0.024315	−0.362966	0.848655	0.559293
	b	0.024315	−0.362966	0.848655	0.559293
	c	0.024315	−0.362966	0.848654	0.559292
	d	0.024315	−0.362966	0.848655	0.559293
	e	0.024315	−0.362966	0.848655	0.559293
	f	0.024315	−0.362966	0.848655	0.559293
N_t	a	−0.002603	0.120798	0.522783	−0.169776
	b	−0.002603	0.120798	0.522783	−0.169776
	c	−0.002603	0.120798	0.522783	−0.169777
	d	−0.002603	0.120798	0.522783	−0.169776
	e	−0.002603	0.120798	0.522783	−0.169776
	f	−0.002603	0.120798	0.522783	−0.169776
w_t	a	0.074057	−0.309272	−1.338454	2.805267
	b	0.074057	−0.309272	−1.338454	2.805267
	c	0.074057	−0.309272	−1.338454	2.805267
	d	0.074057	−0.309272	−1.338454	2.805267
	e	0.074057	−0.309272	−1.338454	2.805267
	f	0.074057	−0.309272	−1.338454	2.805267
q_t	a	−0.009411	−4.984928	11.655303	7.681249
	b	−0.009411	−4.984928	11.655303	7.681249
	c	−0.009412	−4.984927	11.655299	7.681238
	d	−0.009411	−4.984928	11.655303	7.681249
	e	−0.009411	−4.984928	11.655303	7.681249
	f	−0.009411	−4.984928	11.655303	7.681249
Λ_t	a	−1133.179054	41891.655036	−95887.499799	−66989.662742
	b	−1133.179054	41891.655036	−95887.499799	−66989.662742
	c	−1133.182166	41891.667775	−95887.526439	−66989.730945
	d	−1133.179054	41891.655036	−95887.499799	−66989.662742
	e	−1133.179053	41891.655034	−95887.499797	−66989.662730
	f	−1133.179054	41891.655036	−95887.499799	−66989.662742

Notes: See Table A.2.

Table A.4 presents the eigenvalues of the matrix L^x which determines the dynamics of the state variables. While the eigenvalues do not differ in the case $N = 1/3$, the differences in the coefficients of the policy functions are also reflected in different eigenvalues if $N = 0.13$.

Table A.4

Eigenvalues of L^x

Solution	$N = 1/3$	$N = 0.13$
a	0.36592	0.00000
	0.81235	0.81575
	0.99316	0.99245
b	0.36592	0.36592
	0.81235	0.99316
	0.99316	0.81235
c	0.36592	0.52775
	0.81235	0.82000
	0.99316	0.99118
d	0.36592	0.36592
	0.81235	0.81235
	0.99316	0.99316
e	0.36592	0.36592
	0.81235	0.81235
	0.99316	0.99316
f	0.36592	0.36592
	0.81235	0.81235
	0.99316	0.99316

Notes: See Table A.2.