Appendix to

On the Uniqueness of Solutions to Rational Expectations Models

Christopher Heiberger[a], Torben Klarl[b], Alfred Maußner[c]

[a]University of Augsburg, Department of Economics, Universitätsstraße 16, 86159 Augsburg, Germany, christopher.heiberger@wiwi.uni-augsburg.de
[b]University of Augsburg, Department of Economics, Universitätsstraße 16, 86159 Augsburg, Germany, torben.alexander.klarl@wiwi.uni-augsburg.de
[c]Corresponding author, University of Augsburg, Department of Economics, Universitätsstraße 16, 86159 Augsburg, Germany, alfred.maussner@wiwi.uni-augsburg.de

October 19, 2015

Abstract

This document presents steps of the derivation of the linear policy functions based on the Schur decomposition, describes in more detail our model, and presents additional information of the model's solution.

1 Introduction

This Appendix to our paper covers in Section 2 the steps employed to find the linear policy functions of a DSGE model which has been reduced to a minimum number of variables. Section 3 provides the set of equations which determine the dynamics of our example model, and Section 4 presents tables with the policy functions from six different solutions and several measures of the accuracy of the linear solutions.

Equation numbers without the prefix "A" refer to equations of the published paper.

2 Schur Factorization

The (simple) Schur factorization of the matrix W in equation (2.11) is given by

$$S = Z^H W Z, \tag{A.1}$$

where S is an upper triangular matrix with the eigenvalues of W on the main diagonal. Assume that $n(w) = n(x) + n(z)$ eigenvalues are within and $n(v)$ eigenvalues outside the unit circle. S and Z can be chosen so that the first $n(w)$ eigenvalues appear first on the main diagonal of S. In the new variables

$$\begin{bmatrix} \mathbf{w}_t \\ \mathbf{v}_t \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{w}}_t \\ \tilde{\mathbf{v}}_t \end{bmatrix} \tag{A.2}$$

the transformed system reads

$$\mathbb{E}_t \begin{bmatrix} \tilde{\mathbf{w}}_{t+1} \\ \tilde{\mathbf{v}}_{t+1} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ 0_{n(y)\times n(w)} & S_{22} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{w}}_t \\ \tilde{\mathbf{v}}_t \end{bmatrix}. \tag{A.3}$$

Accordingly, the system $\mathbb{E}_t \tilde{\mathbf{v}}_{t+1} = S_{22} \tilde{\mathbf{v}}_t$ is unstable and we must set $\tilde{\mathbf{v}}_t = \mathbf{0}_{n(v)} \forall t$ so that the solution of the linear model (2.11) is

$$L^w = \begin{bmatrix} L_x^x & L_z^x \\ 0_{n(z)\times n(x)} & \Pi \end{bmatrix} = Z_{11} S_{11} Z_{11}^{-1}, \tag{A.4a}$$

$$L^v = \begin{bmatrix} L_x^v & L_z^v \end{bmatrix} = Z_{21} Z_{11}^{-1}. \tag{A.4b}$$

Using (A.4b) in (2.10a) yields

$$L^u = \begin{bmatrix} L_x^u & L_z^u \end{bmatrix} = C_u^{-1} C_{wv} \begin{bmatrix} I_{n(w)} \\ Z_{21} Z_{11}^{-1} \end{bmatrix} \tag{A.4c}$$

so that the matrices from (2.9b) are given by

$$L_x^y = \begin{bmatrix} L_x^u \\ L_x^v \end{bmatrix},\; L_z^y = \begin{bmatrix} L_z^u \\ L_z^v \end{bmatrix}. \tag{A.4d}$$

Apart from numerical imprecision, the solution (A.4) does not differ from (2.9). If we remove the first $n(u)$ equations from the system (2.4) the matrix A can be inverted and the QZ-factorization of $(B - \lambda A)$ delivers the same eigenvalues as the Schur decomposition of W.

3 The Model

3.1 Equilibrium Conditions.

The first-order conditions of maximizing (3.2) subject to (3.1) and given initial values of S_t and B_t are:

$$\Lambda_t = (C_t - \chi^C C_{t-1})^{-\eta}, \tag{A.5a}$$
$$\Lambda_t w_t = \nu_0 (N_t - \chi^N N_{t-1})^{\nu_1}, \tag{A.5b}$$
$$v_t^e = \beta \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t}(d_{t+1} + v_{t+1}^e), \tag{A.5c}$$
$$v_t^b = \beta \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t}, \tag{A.5d}$$

where Λ_t is the Lagrange multiplier of the budget constraint (3.1).

The first-order conditions of the firm's problem – maximizing (3.9) subject to (3.3) and (3.7) and a given initial stock of capital K_t – are

$$w_t = (1-\alpha) Z_t N_t^{-\alpha} K_t^{\alpha}, \tag{A.6a}$$
$$q_t = \frac{1}{\Phi'(I_t/K_t)}, \tag{A.6b}$$
$$q_t \varrho_t = \mathbb{E}_t \varrho_{t+1} \Big\{ \alpha Z_{t+1} N_{t+1}^{1-\alpha} K_{t+1}^{\alpha-1} - (I_{t+1}/K_{t+1}) + q_{t+1} \big[\Phi(I_{t+1}/K_{t+1}) + 1 - \delta \big] \Big\}. \tag{A.6c}$$

In equilibrium all markets clear. We assume that bonds are in zero supply, $B_t = 0 \forall t$, and make no assumption with respect the dividend policy. Accordingly, we disregard d_t and v_t^e. Using equations (3.5) and (3.6) the household's budget constraint (3.1) reduces to the economy's resource restriction $Y_t = C_t + I_t$. Equilibrium in the market for shares requires

$$\varrho_{t+s} = \beta^s \frac{\Lambda_{t+s}}{\Lambda_t}.$$

Let $\mathbf{x}_t = [K_t, C_{t-1}, N_{t-1}]'$, $\mathbf{y}_t := [Y_t, C_t, I_t, N_t, w_t, q_t, \Lambda_t]'$, $\mathbf{z}_t := \ln Z_t$. Then, the system (2.1) is given by:

$$\Lambda_t = (C_t - \chi^C C_{t-1})^{-\eta}, \tag{A.7a}$$
$$\Lambda_t w_t = \nu_0 (N_t - \chi^N N_{t-1})^{\nu_1}, \tag{A.7b}$$
$$w_t = (1-\alpha) Z_t N_t^{-\alpha} K_t^{\alpha}, \tag{A.7c}$$
$$q_t = \frac{1}{\Phi'(I_t/K_t)}, \tag{A.7d}$$
$$Y_t = Z_t N_t^{1-\alpha} K_t^{\alpha}, \tag{A.7e}$$
$$Y_t = C_t + I_t, \tag{A.7f}$$
$$q_t = \beta \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda t} \Big\{ \alpha Z_{t+1} N_{t+1}^{1-\alpha} K_{t+1}^{\alpha-1} - (I_{t+1}/K_{t+1}) + q_{t+1} \big[\Phi(I_{t+1}/K_{t+1}) + 1 - \delta \big] \Big\} \tag{A.7g}$$

$$K_{t+1} = \Phi(I_t/K_t)K_t + (1-\delta)K_t. \tag{A.7h}$$

Note that equations (A.7a)-(A.7f) involve only variables dated at t (using the definition of $\mathbf{x}_t$ above). Therefore, the matrix A of the linearized model will be singular.

3.2 Deterministic Stationary Equilibrium.

Assume $\sigma = 0$ so that $\ln Z_t$ equals its unconditional expectation 0 for all t (and, hence, $Z = 1$). In this case, we can ignore the expectations operator $\mathbb{E}_t$. Stationarity implies $x_{t+1} = x_t = x$ for any variable in our model. As usual, we specify Φ so that adjustment costs play no role in the stationary equilibrium, i.e., $\Phi(I/K)K = \delta K$ and $q = \Phi'(\delta) = 1$. This requires that we choose

$$a_1 = \delta^{\zeta}, \tag{A.8a}$$

$$a_2 = \frac{-\zeta\delta}{1-\zeta}. \tag{A.8b}$$

These assumptions imply via equation (A.7g) the stationary solution for the output-capital-ratio:

$$\frac{Y}{K} = \frac{1-\beta(1-\delta)}{\alpha\beta}. \tag{A.9a}$$

Using the production function, we can solve for the capital-labor ratio and for labor productivity:

$$\frac{K}{N} = \left(\frac{Y}{K}\right)^{\frac{1}{\alpha-1}}, \tag{A.9b}$$

$$\frac{Y}{N} = \left(\frac{Y}{K}\right)^{\frac{\alpha}{\alpha-1}}. \tag{A.9c}$$

Given these solutions equations (A.7a)-(A.7c) and (A.7e) can be reduced to an equation in N:

$$(1-\alpha)\frac{Y}{N} = \nu_0(1-\chi^N)^{\nu_1}(1-\chi^C)^{\eta}\left(\frac{Y}{N} - \delta\frac{K}{N}\right)^{\eta} N^{\nu_1+\eta}. \tag{A.9d}$$

We solve this equation for the parameter ν_0 given our calibration target for N. The levels of the stock of capital K, output Y, consumption C, and investment I can be computed from (A.9a)-(A.9c). In the final step, equation (A.7a) delivers the stationary level of the Lagrange multiplier Λ.

Table A.1 presents the values of the model's variables for both values of N. The values of the parameter ν_0 is determined from equation (A.7b) so as to imply the given N.

The smaller value of $N = 0.13$ implies smaller values of output Y, consumption C, investment I, and the stock of capital K. Accordingly, the value of Λ computed from (A.7a) is much larger. The values of the parameters of the capital adjustment function Φ defined in equation (3.8) are independent of the value of N (see equations (A.8)).

Table A.1: Stationary Solution

Variable	$N = 1/3$	$N = 0.13$
Y	$0.123469E{+}01$	$0.481528E{+}00$
C	$0.918109E{+}00$	$0.358063E{+}00$
I	$0.316577E{+}00$	$0.123465E{+}00$
K	$0.126631E{+}02$	$0.493860E{+}01$
w	$0.237060E{+}01$	$0.237060E{+}01$
q	$0.100000E{+}01$	$0.100000E{+}01$
Λ	$0.811271E{+}04$	$0.899172E{+}06$
ν_0	$0.218095E{+}08$	$0.254484E{+}11$
a_1	$0.108273E{-}06$	$0.108273E{-}06$
a_2	$0.324675E{-}01$	$0.324675E{-}01$

3.3 The Loglinear System.

Let $\hat{x}_t \equiv (x_t - x)/x \simeq \ln(x_t/x)$ denote the percentage deviation of x_t from its stationary solution x. In terms of this variables the linearized system (A.7b) reads:

$$\hat{\Lambda}_t = \frac{-\eta}{1-\chi^C}\hat{C}_t + \frac{\eta\chi^C}{1-\chi^C}\hat{C}_{t-1}, \tag{A.10a}$$

$$\hat{\Lambda}_t = -\hat{w}_t + \frac{\nu_1}{1-\chi^N}\hat{N}_t - \frac{\nu_1\chi^N}{1-\chi^N}\hat{N}_{t-1}, \tag{A.10b}$$

$$\hat{w}_t = \hat{Z}_t - \alpha\hat{N}_t + \alpha\hat{K}_t, \tag{A.10c}$$

$$\hat{q}_t = \zeta\hat{I}_t - \zeta\hat{K}_t, \tag{A.10d}$$

$$\hat{Y}_t = \hat{Z}_t + (1-\alpha)\hat{N}_t + \alpha\hat{K}_t, \tag{A.10e}$$

$$\hat{Y}_t = \frac{C}{Y}\hat{C}_t + \frac{I}{Y}\hat{I}_t, \tag{A.10f}$$

$$\hat{q}_t = \mathbb{E}_t\hat{\Lambda}_{t+1} - \hat{\Lambda}_t + \beta\mathbb{E}_t\hat{q}_{t+1} + \alpha\beta\frac{Y}{K}\mathbb{E}_t\left[\hat{Z}_{t+1} + (1-\alpha)\hat{N}_{t+1} + (\alpha-1)\hat{K}_{t+1}\right], \tag{A.10g}$$

$$\hat{K}_{t+1} = (1-\delta)\hat{K}_t + \delta\hat{I}_t. \tag{A.10h}$$

None of the coefficients in this system depends on the level of hours N: According to (A.9a), Y/K is a function of the parameters α, β, and δ. The variable I/K is equal to $\delta/(Y/K)$ and C/Y is equal to $C/Y = (C/K)/(Y/K) = (1 - \delta/(Y/K))$.

4 Numerical Results

Table A.2 and A.3 display the coefficients of the policy functions for $N = 0.13$ and $N = 1/3$, respectively. The entries represent the coefficients of the policy functions for the variables listed in the first column. The labels “a”-“f” refer to the different solutions:

a: factorization of the pencil $(B - \lambda A)$, with A and B unbalanced,

b: factorization of the pencil $(B - \lambda A)$, with A and B balanced,

c: factorization of the pencil $(A - \mu B)$, with A and B unbalanced,

d: factorization of the pencil $(A - \mu B)$, with A and B balanced,

e: solution of the reduced system,

f: solution of the non-linear system.

Table A.2: Policy Functions: $N = 0.13$

Dependent Variables		Independent Variables			
		K_t	C_{t-1}	N_{t-1}	$\ln Z_t$
K_{t+1}	a	0.999760	−0.364790	0.852468	0.221933
	b	0.999315	−0.362966	0.848655	0.218124
	c	0.991181	0.007051	0.000000	0.000000
	d	0.999315	−0.362966	0.848655	0.218124
	e	0.999315	−0.362966	0.848655	0.218124
	f	0.999315	−0.362966	0.848655	0.218124
Y_t	a	0.029211	0.285220	1.241699	0.326950
	b	0.028931	0.286363	1.239309	0.324564
	c	0.029714	0.280971	1.251072	0.332221
	d	0.028931	0.286363	1.239309	0.324564
	e	0.028931	0.286363	1.239309	0.324564
	f	0.028931	0.286363	1.239309	0.324564
C_t	a	0.004450	0.650011	0.389230	0.105017
	b	0.004617	0.649329	0.390655	0.106440
	c	0.004150	0.652543	0.383644	0.101876
	d	0.004617	0.649329	0.390655	0.106440
	e	0.004617	0.649329	0.390655	0.106440
	f	0.004617	0.649329	0.390655	0.106440
I_t	a	0.024760	−0.364790	0.852468	0.221933
	b	0.024315	−0.362966	0.848655	0.218124
	c	0.025381	−0.364155	0.850436	0.225832
	d	0.024315	−0.362966	0.848655	0.218124
	e	0.024315	−0.362966	0.848655	0.218124
	f	0.024315	−0.362966	0.848655	0.218124
N_t	a	−0.002485	0.120316	0.523791	−0.065206
	b	−0.002603	0.120798	0.522783	−0.066213
	c	−0.002272	0.118523	0.527745	−0.062983
	d	−0.002603	0.120798	0.522783	−0.066213
	e	−0.002603	0.120798	0.522783	−0.066213
	f	−0.002603	0.120798	0.522783	−0.066213
w_t	a	0.189117	−0.789841	−3.438550	2.798658
	b	0.189890	−0.793006	−3.431933	2.805267
	c	0.187722	−0.778074	−3.464506	2.784063
	d	0.189890	−0.793006	−3.431933	2.805267
	e	0.189890	−0.793006	−3.431933	2.805267

Table A.2 continued

Dependent Variables		Independent Variables			
		K_t	C_{t-1}	N_{t-1}	$\ln Z_t$
	f	0.189890	−0.793006	−3.431933	2.805267
q_t	a	−0.008443	−12.846094	30.019700	7.815374
	b	−0.024132	−12.781868	29.885393	7.681249
	c	0.013659	−12.834069	29.971782	7.958970
	d	−0.024132	−12.781868	29.885393	7.681249
	e	−0.024132	−12.781868	29.885393	7.681249
	f	−0.024132	−12.781868	29.885393	7.681249
Λ_t	a	−310435.585054	11857778.135886	−27151143.432483	−7325585.987312
	b	−322040.799510	11905287.194025	−27250492.309927	−7424799.941246
	c	−289487.870911	11681115.175497	−26761474.970035	−7106477.042567
	d	−322040.799510	11905287.194025	−27250492.309927	−7424799.941246
	e	−322040.809449	11905287.238673	−27250492.350733	−7424800.029568
	f	−322040.799510	11905287.194025	−27250492.309927	−7424799.941246

Table A.3: Policy Functions: $N = 1/3$

Dependent Variables		Independent Variables			
		K_t	C_{t-1}	N_{t-1}	$\ln Z_t$
K_{t+1}	a	0.999315	−0.362966	0.848655	0.559293
	b	0.999315	−0.362966	0.848655	0.559293
	c	0.999315	−0.362968	0.848658	0.559294
	d	0.999315	−0.362966	0.848655	0.559293
	e	0.999315	−0.362966	0.848655	0.559293
	f	0.999315	−0.362966	0.848655	0.559293
Y_t	a	0.028931	0.286363	1.239309	0.832215
	b	0.028931	0.286363	1.239309	0.832215
	c	0.028931	0.286363	1.239309	0.832214
	d	0.028931	0.286363	1.239309	0.832215
	e	0.028931	0.286363	1.239309	0.832215
	f	0.028931	0.286363	1.239309	0.832215
C_t	a	0.004617	0.649329	0.390655	0.272922
	b	0.004617	0.649329	0.390655	0.272922
	c	0.004617	0.649329	0.390655	0.272922
	d	0.004617	0.649329	0.390655	0.272922
	e	0.004617	0.649329	0.390655	0.272922
	f	0.004617	0.649329	0.390655	0.272922
I_t	a	0.024315	−0.362966	0.848655	0.559293
	b	0.024315	−0.362966	0.848655	0.559293
	c	0.024315	−0.362966	0.848654	0.559292
	d	0.024315	−0.362966	0.848655	0.559293
	e	0.024315	−0.362966	0.848655	0.559293
	f	0.024315	−0.362966	0.848655	0.559293
N_t	a	−0.002603	0.120798	0.522783	−0.169776
	b	−0.002603	0.120798	0.522783	−0.169776
	c	−0.002603	0.120798	0.522783	−0.169777
	d	−0.002603	0.120798	0.522783	−0.169776

Table A.3 continued

Dependent Variables		Independent Variables			
		K_t	C_{t-1}	N_{t-1}	$\ln Z_t$
	e	−0.002603	0.120798	0.522783	−0.169776
	f	−0.002603	0.120798	0.522783	−0.169776
w_t	a	0.074057	−0.309272	−1.338454	2.805267
	b	0.074057	−0.309272	−1.338454	2.805267
	c	0.074057	−0.309272	−1.338454	2.805267
	d	0.074057	−0.309272	−1.338454	2.805267
	e	0.074057	−0.309272	−1.338454	2.805267
	f	0.074057	−0.309272	−1.338454	2.805267
q_t	a	−0.009411	−4.984928	11.655303	7.681249
	b	−0.009411	−4.984928	11.655303	7.681249
	c	−0.009412	−4.984927	11.655299	7.681238
	d	−0.009411	−4.984928	11.655303	7.681249
	e	−0.009411	−4.984928	11.655303	7.681249
	f	−0.009411	−4.984928	11.655303	7.681249
Λ_t	a	−1133.179054	41891.655036	−95887.499799	−66989.662742
	b	−1133.179054	41891.655036	−95887.499799	−66989.662742
	c	−1133.182166	41891.667775	−95887.526439	−66989.730945
	d	−1133.179054	41891.655036	−95887.499799	−66989.662742
	e	−1133.179053	41891.655034	−95887.499797	−66989.662730
	f	−1133.179054	41891.655036	−95887.499799	−66989.662742

Table A.4 presents the eigenvalues of the matrix L^x which determines the dynamics of the state variables. While the eigenvalues do not differ in the case $N = 1/3$, the differences in the coefficients of the policy functions are also reflected in different eigenvalues if $N = 0.13$.

Table A.4: Eigenvalues of L^x

Solution	$N = 1/3$	$N = 0.13$
a	0.36592	0.00000
	0.81235	0.81575
	0.99316	0.99245
b	0.36592	0.36592
	0.81235	0.99316
	0.99316	0.81235
c	0.36592	0.52775
	0.81235	0.82000
	0.99316	0.99118
d	0.36592	0.36592
	0.81235	0.81235
	0.99316	0.99316
e	0.36592	0.36592
	0.81235	0.81235
	0.99316	0.99316
f	0.36592	0.36592
	0.81235	0.81235
	0.99316	0.99316